MEAN SLIPPAGE PROBLEMS

by

BU-804-M Revised September, 1983

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Abstract

Mean slippage is defined and a variety of mean slippage problems are described, including both slippage of a single population and multiple slippage. Nonparametric and parametric formulations are treated. The connection between slippage and outliers is discussed.

Paper No. BU-804-M in the Biometrics Unit series.

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Encyclo. Statist. Sci.

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MEAN SLIPPAGE PROBLEMS

In a collection of populations, mean slippage occurs when one or several of the population means differ from the common mean of the remaining populations. Any populations whose means deviate from this common mean are said to have slipped. A slippage test is a rule for determining whether slippage has occurred and identifying which populations, if any, have slipped. The study of slippage problems centers on the search for rules that perform these tasks well.

The mean slippage framework encompasses a wide range of situations. The populations may be normal* or nonnormal (e.g., gamma), or a nonparametric* approach may be followed. The model of interest may be either a single slippage, in which the number of slipped populations is known to be at most one, or multiple slippage, in which there is the possibility of several slipped populations; in the latter case, the slipped means may be equal or unequal among themselves. A control group, known not to have slipped, may be either present or absent. The direction of mean slippage may be known, as when any slippage that occurs must be positive, or unknown, as when either positive or negative slippage is possible. Sample sizes from the populations may be equal or unequal. Observations may be univariate or multivariate. This list, though not exhaustive, illustrates the diversity of mean slippage problems. Fortunately, many of these problems can be treated by a common approach.

Mean slippage was first studied by Mosteller [12]. He considered equal-sized random samples from n continuous populations, with the null hypothesis that all n populations are identical and the alternative that one among them has slipped to the right, the rest remaining identical. His rule is to find the sample containing the largest observation, determine how many observations in this sample exceed all observations in all other samples, and reject the null hypothesis when this number is sufficiently large. Another rule, due to Doornbos and Prins [5] and likely to be more powerful than Mosteller's rule [9], is to reject the null hypothesis if the greatest sample rank sum is sufficiently large, where the ith sample rank sum is the sum of the overall ranks of the observations from the ith sample.

To illustrate these methods, consider the following data on seven varieties of guayule, with five observations of rubber yield per variety:

Variety	Observations					Mean
1	12.15	8.20	8.94	12.27	7.32	9.776
2	12.19	4.09	8.86	8.26	7.44	8.168
3	10.54	11.71	13.90	4.96	8.51	9.924
4	7.18	9.29	5.32	5.67	5.94	6.680
5	11.82	9.88	12.62	2.88	7.34	8.908
6	14.33	9.80	12.89	13.72	17.55	13.658
7	8.21	9.08	9.90	6.62	9.10	8.582

Each observation represents the rubber yield in grams obtained from two plants randomly selected in a plot. To test for slippage of one population to the right with Mosteller's rule, observe that variety 6 has the largest observation, 17.55. The number r of observations of variety 6 that exceed all observations in other samples (17.55, 14.33) is two. For seven samples of size five, $\Pr[r \geq 2] = .118$ when all populations are identical (see [1, Sec. 5.1.1]), so the null hypothesis is not rejected at the $\alpha = .05$ level. (Mosteller's rule would reject the null hypothesis if the 13.90 observed for variety 3 were 13.70, resulting in $r = 3$, because $\Pr[r \geq 3] = .0107$ under the null hypothesis.) Variety 6 has the greatest sample rank sum of 34 + 21 + 31 + 32 + 35 = 153. This exceeds the $\alpha = .01$ critical value of 150 (see [9, Appendix 7]), so the rule of Doornbos and Prins rejects the null hypothesis at this level. (The data are from Federer [6, p. 122], but each observation x has been replaced here by 20 - x, e.g., 7.85 by 12.15, 11.80 by 8.20, for consistency with the discussion above of slippage to the right. With obvious adjustments, these methods could be used to examine the original data for slippage to the left.)

For extensions, modifications, and competitors of these nonparametric rules, see Barnett and Lewis [1], David [3], and Hawkins [9]. Multiple slippage, unequal sample sizes, and other cases are treated. Hashemi-Parast and Young [8] dealt with distribution free procedures based on sample linear rank statistics, in particular on exponential scores. Neave [13] discussed several quick, simple tests based on extreme observations. Joshi and Sathe [10] proposed another test based on extreme observations.

The earliest work on a parametric mean slippage model was by Paulson [14]. He took a multiple decision approach to the single slippage problem. The mutually independent $N(\mu_i, \sigma^2)$ random variables X_{ij} $(i = 1, \cdots, n;\ j = 1, \cdots, m)$ form n random samples of equal size m. Let D_0 be the decision that all of the means are equal, and D_i $(i = 1, \cdots, n)$ the decision that population i has slipped to the right, that is, $\mu_i = \mu + \delta$ where μ is the mean of each of the $n-1$ populations other than the ith and $\delta > 0$. Statistical rules that choose optimally, in some sense, among these $n+1$ decisions are desired. Three reasonable restrictions on a rule are (i) when all means are equal, making D_0 correct, the probability of selecting D_0 is $1-\alpha$, (ii) the rule is invariant* under the transformation $y = ax + b$ of the observations, where $a > 0$ and b are constants, and (iii) the rule is symmetric, that is, the probability of selecting D_i when population i has slipped is the same for every i. Under these restrictions, the probability of making the correct decision when one population has slipped to the right is maximized by the rule: compute

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$$T = m(\bar{x}_M - \bar{x})\Big/\left[\sum_{i=1}^{n}\sum_{j=1}^{m}(x_{ij} - \bar{x})^2\right]^{\frac{1}{2}},$$

where $\bar{x}$ is the mean of all mn observations, and the sample mean $\bar{x}_M$ from population M is the largest of the n sample means; if $T \leq c_\alpha$, select D_0; if $T > c_\alpha$, select D_M, where the constant c_α is chosen to make $P[T > c_\alpha] = \alpha$ when all means are equal.

Returning to the guayule data, it is routine to calculate $T = 5(13.658 - 9.385)/(352.57)^{\frac{1}{2}} = 1.138$. This is greater than the $\alpha = .01$ critical value of 1.01 (see [1, Sec. 5.3.1]), so Paulson's rule rejects the null hypothesis at this level, concluding that variety 6 (= M) has slipped.

This rule has been modified for use with slippage in an unspecified direction, additional external information about the variance σ^2, known variance, and unequal sample sizes. Rules for nonnormal populations with gamma, Poisson, binomial, and other distributions are discussed in Doornbos [4]. For details on these and related matters, see references [1] and [3].

A Bayesian treatment of slippage problems was given by Karlin and Truax [11]. For the case of a single slippage, they derived optimal rules under very general conditions by characterizing the class of Bayes rules (see BAYESIAN INFERENCE) within the set of all rules that obey certain natural restrictions of invariance and symmetry, and then showing that Bayes procedures are uniformly most powerful*. Many special cases were examined in detail, including nonparametric situations, Paulson's model, multivariate observations, and the presence of a control group.

There is a close connection between slippage and outliers*, since a slippage problem with one observation from each population can be formulated as an outlier problem, with each slipped population corresponding to an outlier. For example, let $X_1, \cdots, X_n$ be independent normal observations with variance σ^2, one of which has mean $\mu + \delta$, where $\delta \neq 0$, and the remaining $n-1$ of which have common mean μ. This is a mean slippage model with one slipped population. The outlier literature refers to this situation as model A, and to the observation from the slipped population as an outlier caused by mean slippage. Thus outlier results for model A apply immediately to mean slippage problems with normal populations and samples of equal size. Schwager and Margolin [15] treated a problem of this type with an unknown number of outliers.

Under the multiple slippage model, several population means deviate from the common mean of the rest. For example, if there are n populations, the distribution of population i is $N(\mu_i, \sigma^2)$, $n-2$ of the means μ_i have the common value μ, and the remaining two means have the values $\mu + \delta_1$

and $\mu + \delta_2$, where $\delta_1 > 0$ and $\delta_2 > 0$, then two populations have slipped, possibly by differing amounts. When the number k of slipped populations is fixed, the multiple decision approach has the null hypothesis of no slippage and $\binom{n}{k}$ slippage alternatives that some unknown set of k populations differ from the remaining n - k. Butler [2] and Singh [16] treated this situation, which has also been addressed in the outlier literature as model A with multiple outliers.

References

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(DECISION THEORY

OUTLIERS

VARIANCE SLIPPAGE PROBLEMS)